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A convergence–divergence test for series of nonnegative terms

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ABSTRACT

We present a convergence–divergence test for series of nonnegative terms. Our proof is elementary, and yet we show examples of application to some apparently difficult cases.

This short note starts from two results, at first glance perhaps surprising:

$$\text{(i)} \sum_{n=1}^{\infty} \frac{1}{n^{2+\cos n}} = \infty, \qquad \text{(ii)} \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{2+\cos n}{3}\right)^{n} < \infty.$$

Case (i) has been recently addressed, in [1], where the authors give a proof that was (in their own words) at the frontier between analysis and number theory. Case (ii) apparently originated in a curious way: it was proposed in a calculus exam by mistake, and remained open for a long time thereafter. A solution was devised only ten years later [5], once again by means of quite sophisticated tools.

We give here a sort of condensation method applicable, in principle, to any series of nonnegative terms

$$\sum_{n=1}^{\infty} a_n \tag{S}$$

even when the sequence a_n fails to be monotonic. We subsequently apply the result to the above-mentioned examples, showing that there is no need to invoke deep number-theoretic properties of π. Indeed, irrationality alone suffices for (i), whereas for (ii) it is enough to know that π is not a Liouville number, i.e. its irrationality measure is finite, no matter how large.

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1. The abstract result

Notation. The symbols n, m, p, κ will denote elements of $\mathbb{N} = \{1, 2, \ldots\}$. Closed intervals are taken in $\mathbb{N}$ as well. Finally, c_n is a given sequence satisfying $\sum_{n=1}^{\infty} c_n < \infty$.

Theorem. *The following convergence–divergence test holds.*

(c) *The series (S) converges if $\{na_n\}$ is a bounded sequence and there exist $\varrho, \vartheta \geq 0$ and $\varepsilon \in (0, 1]$ such that*

$$\operatorname{card}\{p \in [1, m] : a_{n+p} > c_n\} \leq \varrho m^{1-\varepsilon}$$

for every m large and every $n \geq m^{\vartheta}$.

(d) *The series (S) diverges if there exist $\omega > 0$ and $\lambda \geq 0$ such that the inequality*

$$\max_{p\in[1,m]} a_{\kappa m+p} \geq \frac{\omega}{(\kappa m + m)^{1+\lambda/m}}$$

holds for infinitely many m and every κ.

Proof of (c). Without loss of generality, we can assume that

$$a_n \leq 1/n, \qquad \varrho = 1, \qquad \varepsilon < 1, \qquad \vartheta \in \mathbb{N}.$$

Then, for every $m \gg 1$ and every $\kappa \geq m^{\vartheta-1}$, the quantities

$$A(\kappa, m) = \sum_{p=1}^{m} a_{\kappa m+p} \quad \text{and} \quad C(\kappa, m) = \sum_{p=1}^{m} c_{\kappa m+p}$$

satisfy the inequality

$$A(\kappa, m) \leq C(\kappa, m) + \sum_{p=1}^{\lfloor m^{1-\varepsilon}\rfloor} \frac{1}{\kappa m + p} \leq C(\kappa, m) + \frac{1}{\kappa m^{\varepsilon}}.$$

Choosing $m_1 \gg 1$, we define by recursion the sequence

$$m_{\imath+1} = m_\imath \left\lfloor \frac{1}{m_\imath} \exp m_\imath^{\varepsilon/2} \right\rfloor,$$

which is easily seen to be strictly increasing. Furthermore,

$$m_\imath > \imath^{4/\varepsilon}, \quad \forall \imath \in \mathbb{N}.$$

Indeed, given any $\imath_\star \gg 1$, the above inequality is clearly true for every $\imath < \imath_\star$, up to taking m_1 suitably large. Besides this, if $m_\imath > \imath^{4/\varepsilon}$ for some $\imath \geq \imath_\star \gg 1$, we get

$$m_{\imath+1} \geq \exp m_\imath^{\varepsilon/2} - m_\imath > \exp \imath^2 - \imath^{4/\varepsilon} > (\imath + 1)^{4/\varepsilon}.$$

Therefore, defining $q = m_1^{\vartheta} + 1$ and considering the intervals

$$\mathbb{K}_\imath = [m_\imath^{\vartheta-1}, m_{\imath+1}^{\vartheta}/m_\imath - 1],$$

we can write (S) in the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{q-1} a_n + \sum_{n=q}^{\infty} a_n = \sum_{n=1}^{q-1} a_n + \sum_{\imath=1}^{\infty} \sum_{\kappa\in\mathbb{K}_\imath} A(\kappa, m_\imath),$$

where, in light of the previous discussion,

$$\sum_{\imath=1}^{\infty} \sum_{\kappa\in\mathbb{K}_\imath} A(\kappa, m_\imath) \leq \sum_{\imath=1}^{\infty} \sum_{\kappa\in\mathbb{K}_\imath} C(\kappa, m_\imath) + \sum_{\imath=1}^{\infty} \sum_{\kappa\in\mathbb{K}_\imath} \frac{1}{\kappa m_\imath^{\varepsilon}}.$$

The first term in the right-hand side reads

$$\sum_{\imath=1}^{\infty}\sum_{\kappa\in\mathbb{K}_\imath} C(\kappa, m_\imath) = \sum_{n=q}^{\infty} c_n < \infty.$$

Concerning the second one, estimating the sum

$$\sum_{\kappa\in\mathbb{K}_\imath}\frac{1}{\kappa m_\imath^{\varepsilon}} \le \frac{1}{m_\imath^{\varepsilon}}\sum_{\kappa=1}^{m_{\imath+1}^{\vartheta}}\frac{1}{\kappa} \le \frac{\vartheta\log m_{\imath+1}}{m_\imath^{\varepsilon}}$$

and observing that $\log m_{\imath+1} \le m_\imath^{\varepsilon/2}$, we end up with the bound

$$\sum_{\imath=1}^{\infty}\sum_{\kappa\in\mathbb{K}_\imath}\frac{1}{\kappa m_\imath^{\varepsilon}} \le \sum_{\imath=1}^{\infty}\frac{\vartheta}{m_\imath^{\varepsilon/2}} \le \sum_{\imath=1}^{\infty}\frac{\vartheta}{\imath^2} < \infty. \quad \square$$

Proof of (d). There is no harm in assuming $\lambda > 0$. For any m for which the inequality holds, the *tail* of the series

$$\mathfrak{t}_m = \sum_{n=m+1}^{\infty} a_n = \sum_{\kappa=1}^{\infty}\sum_{p=1}^{m} a_{\kappa m+p}$$

fulfills the following estimate from below:

$$\mathfrak{t}_m \ge \sum_{\kappa=1}^{\infty}\max_{p\in[1,m]} a_{\kappa m+p} \ge \frac{\omega}{m^{1+\lambda/m}}\sum_{\kappa=2}^{\infty}\frac{1}{\kappa^{1+\lambda/m}} \ge \frac{\omega}{\lambda(2m)^{\lambda/m}}.$$

Since this occurs for infinitely many m, we reach the conclusion that

$$\liminf_{m\to\infty}\mathfrak{t}_m \ge \omega/\lambda > 0,$$

incompatible with the convergence of (S). $\square$

2. Application to (i) and (ii)

For $\alpha \in \mathbb{R}$, the distance between $\hat{\alpha} = \alpha\text{-mod}2\pi$ and 0 on the torus $[0, 2\pi)$ is given by

$$\|\alpha\| := \min\{\hat{\alpha}, 2\pi - \hat{\alpha}\} = \min_{z\in\mathbb{Z}}|\alpha - 2\pi z|.$$

Two facts will be needed: the first one depends only on the irrationality of π, whereas the second is a consequence of π not being a Liouville number (see Appendix).

F1. There are infinitely many m for which

$$\{p \in [1, m] : \|\alpha + p\| \le 2\pi/m\} \ne \emptyset, \quad \forall\alpha \in \mathbb{R}.$$

F2. There exist $\ell \ge 1$ and $\nu > 0$ such that

$$\|n\| > \nu\, n^{-\ell}, \quad \forall n \in \mathbb{N}.$$

Proof of (i). Defining

$$a_n = n^{-2-\cos n},$$

select any $m \ge 4\pi^2$ complying with F1. Then, for every $s \in \mathbb{N}$, the implications

$$\|s + \pi + p\| < 2\pi/m \Rightarrow \|s + \pi + p\| < 1/\sqrt{m} \Rightarrow 1 + \cos(s + p) < 1/m$$

hold for some $p = p(s) \in [1, m]$. Accordingly,

$$a_{s+p} \geq \frac{1}{(s+p)^{1+1/m}} \geq \frac{1}{(s+m)^{1+1/m}}.$$

Choosing $s = \kappa m$, we meet the hypotheses of the divergence test (d) with $\omega = \lambda = 1$. $\square$

Proof of (ii). Consider the sequences

$$a_n = 3^{-n}(2+\cos n)^n n^{-1} \leq n^{-1} \quad \text{and} \quad c_n = n^{-2}.$$

When $n \geq m^{1+2\ell} \gg 1$ and $p \in [1, m]$, it is not hard to verify that

$$a_{n+p} > c_n \Rightarrow \|n+p\| \leq \nu/2m^{\ell}.$$

Consequently, there is at most one $p \in [1, m]$ for which $a_{n+p} > c_n$ for n, m fixed. Indeed, whenever p_1, p_2 verify the inequality, we deduce the estimate

$$\|p_1 - p_2\| \leq \nu m^{-\ell}.$$

At the same time, if $p_1 \neq p_2$, we learn from F2 that

$$\|p_1 - p_2\| \geq \nu |p_1 - p_2|^{-\ell} > \nu m^{-\ell}.$$

In conclusion, the convergence test (c) applies with $\vartheta = 1 + 2\ell$ and $\varrho = \varepsilon = 1$. $\square$

Remark. In neither case is the general theorem used in its full strength. We may also observe that both results remain the same (as well as the proofs) when $\cos n$ is replaced by $\cos(n + \phi)$ with $\phi \in [0, 2\pi)$.

Appendix

For the reader's convenience, we sketch the proofs of F1 and F2.

Proof of F1. Given an irrational number $x > 0$, the celebrated Dirichlet approximation theorem[1] ensures the existence of infinitely many irreducible fractions $r = n/m$ satisfying

$$0 < |1 - xr| < x/m^2.$$

Choosing $x = 2\pi$, select any of those $r = n/m$ and suppose that $2\pi r < 1$ (the other case being completely analogous). Then, for an arbitrarily fixed $\alpha \in \mathbb{R}$, there is $q \in [1, m]$ such that

$$2\pi - \hat{\alpha} \in I_q := \left[\frac{2\pi(q-1)}{m}, \frac{2\pi q}{m}\right).$$

Making use of the straightforward identity

$$\|\alpha + p\| = \|\hat{p} - (2\pi - \hat{\alpha})\|,$$

the claim follows by exhibiting $p \in [1, m]$ with $\hat{p} \in I_q$. Indeed, for every $p \in [1, m]$, the above inequality yields

$$\frac{2\pi(q-1)}{m} < p - \frac{2\pi(np + 1 - q)}{m} < \frac{2\pi p(1 + mq - m)}{m^2} \leq \frac{2\pi q}{m}.$$

Since n and m are relatively prime and $q \in [1, m]$, we can find $p \in [1, m]$ such that $np + 1 - q = zm$ for some $z \in \mathbb{Z}$, so obtaining

$$p - 2\pi z \in I_q \Rightarrow z = \lfloor p/2\pi \rfloor \Rightarrow p - 2\pi z = \hat{p}. \quad \square$$

1 A quite direct consequence of the *pigeonhole principle* (see e.g. [2]).

Proof of F2. The irrationality measure of π is by definition the smallest $\mu \geq 2$ such that

$$|\pi - \tau| > m^{-\mu-\delta}, \quad \forall \delta > 0,$$

for all $\tau = z/m \in \mathbb{Q}$ with $m > 0$ large enough. It is well known that $\mu < \infty$, which is the same as saying that π is not a Liouville number.[2] Accordingly, fixing $\ell > \mu - 1$ and choosing $\tau = n/2m$, we get

$$|n - 2\pi m| > \nu m^{-\ell}, \quad \forall n,\ m \in \mathbb{N},$$

for some $\nu > 0$ suitably small. When $m \leq n$, this readily gives

$$|n - 2\pi m| > \nu n^{-\ell}.$$

But such an inequality remains trivially true if $m > n$, where $|n - 2\pi m| > 5$, as well as if we take $m \leq 0$. Summarizing, every $n \in \mathbb{N}$ fulfills the lower bound

$$\|n\| = \min_{z \in \mathbb{Z}} |n - 2\pi z| > \nu n^{-\ell}. \quad \square$$

References

[1] B. Brighi, N. Chevallier, $\sum 1/n^{2+\cos n}$, Revue de la filière Mathématique (RMS) 119 (2008–2009) 3–8.
[2] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, Oxford, 1979.
[3] K. Mahler, On the approximation of π, Nederl. Akad. Wetensch. Proc. Ser. A. 56/Indagationes Math. 15 (1953) 30–42.
[4] V.Kh. Salikhov, On the irrationality measure of π, Russian Math. Surveys 63 (2008) 570–572.
[5] A. Stadler, A calculus exam misprint ten years later, SIAM Problems and Solutions (2009). Available online at: http://www.siam.org/journals/problems.php.

2 The proof is due to Mahler [3]. The best estimate currently available is $\mu \leq 7.6304$ (see [4]).